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On the Asymptotic Behavior of Dynamical Maps for a Finite Quantum System

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Abstract

We give some remarks on the dynamical evolution (also nonlinear) of finite quantum system. We are interested in t-asymptotic behavior of density matrices in the Liouville space formalism and we show that for nonlinear dynamical semigroups, as well as for the dynamical maps that do not form semigroups, the stationary time evolution may be attained for finite time in contrast to the motion generated by the linear dynamical semigroup. Recently the problem of constructing a nonlinear analog of quantum mechanics with nonlinear wave equation playing the role of the Schrödinger equation has been investigated by some authors; see for example Mielnik (1974), Bergmann (1968). Our work is related to this investigation and gives a characteristic feature of the nonlinear time evolution.

We will be dealing with the time evolution of the statistical state of a finite quantum system Σ. The most general dynamical evolution of Σ is given by a one-parameter family $\{\Lambda_t, t \geqslant 0\}$ of transformations (also non-linear) which maps the set of density matrices into itself. If ρ represents the state of the system at time $t = 0$, then $\Lambda_t\rho, t \geqslant 0$ gives the state at the later time t (thus it must be the case that $\Lambda_0 = 1$).

From the physical reasons we assume that the function $t \to P(a, \Lambda_t\rho, E)$ giving the probability that the measurement of an observable a in the state $\Lambda_t\rho$ will give a value lying in a Borel set $E \subset R^1$, is a continuous function of the parameter $t \geqslant 0$ (for any fixed triple (a, ρ, E)).

The mathematical description of the above framework is as follows: Let H be a separable complex Hilbert space corresponding to our system Σ. The set of all density operators (states) will be denoted by $\mathscr{S}(H)$, and the set of all observables by $O(H)$. The expectation value of an observable a in the state ρ is given by $\rho(a) = \mathrm{Tr}\,(\rho a)$. Let $L^1(H)$ be the real Banach space of self-adjoint, trace-class linear operators on H with the trace norm

$$\|\rho\|_1 = \sup_{\{x_n\},\{y_n\}} \sum_{n=1}^{\dim H} |(x_n, \rho y_n)|$$

where supremum is taken over all orthonormal and complete bases $\{x_n\}$, $\{y_n\}$ in H. Space $L^1(H)$ is the smallest linear space that contains the set $\mathscr{S}(H)$.

The one-parameter family $\{\Lambda_t, t \geqslant 0\}$ of transformations $\Lambda_t : L^1(H) \to L^1(H)$ for each $t \geqslant 0$, describes a time evolution of the system Σ if the following conditions are satisfied:

(i) Λ_t: $L^1_+(H) \to L^1_+(H)$; $L^1_+(H)$–the positive cone in $L^1(H)$.

(ii) $\|\Lambda_t \rho\|_1 = \|\rho\|_1$ for $t \geqslant 0$ and $\rho \in L^1_+(H)$.

(iii) $t \to (\Lambda_t\rho)(a)$ is a continuous function of $t \geqslant 0$ for every bounded observable a, and $\rho \in \mathscr{S}(H)$.

(iv) $s - \lim_{t \downarrow 0} \Lambda_t \rho = \rho$.

The statement (iii) ensures the continuity of the probability function $t \to P(a, \Lambda_t \rho, E)$.

It is evident that for the dynamical description of quantum systems the question concerning the existence in the suitable topology of the limits

$$\lim_{t \to t_0} \Lambda_t \rho, \qquad t_0 \in [0, +\infty], \qquad \rho \in L^1_+(H)$$

is very important. Here we will be dealing with so-called w_1 topology. We say that a sequence ρ_n of elements of $L^1(H)$ converges in w_1 topology to an operator $\rho \in B(H)$ if

$$\mathrm{Tr}\,(\rho_n a) \to \mathrm{Tr}\,(\rho a)$$

for all $a \in O(H)$ and we write $w_1 - \lim \rho_n = \rho$. We note that $L^1(H)$ is weakly sequentially complete and thus $\rho \in L^1(H)$.

(A) Let the dynamics of Σ be invariant under time translations. Then $\Lambda_t \cdot \Lambda_s = \Lambda_{t+s}$, $t, s \geqslant 0$, i.e., the maps Λ_t form a semigroup. We also assume that each Λ_t preserves the convex combinations of states:

$$\Lambda_t[\alpha\rho_1 + (1-\alpha)\rho_2] = \alpha\Lambda_t\rho_1 + (1-\alpha)\Lambda_t\rho_2$$

where $0 \leqslant \alpha \leqslant 1$ and $\rho_1, \rho_2 \in \mathscr{S}(H)$.

This assumption is very restrictive since it implies (Guz, 1974) that the semigroup $\{\Lambda_t, t \geqslant 0\}$ is a (C_0)-class contracting semigroup of positive linear operators acting on the partially ordered real Banach space $L^1(H)$. The semigroup $\{\Lambda_t, t \geqslant 0\}$ will be called the dynamical semigroup (Kossakowski, 1972).

(a) Under suitable conditions the dynamical semigroup can be extended to a group. This can be done, for example, when $\|\Lambda_t \rho\|_1 = \|\rho\|_1$ for each $\rho \in L^1(H)$ (Kossakowski, 1972), or when every Λ_t is a rank-preserving (super) operator (Posiewnik, 1976). Then there exists a strongly continuous one-parameter group $\{V(t); t \in \mathbb{R}^1\}$ of unitary operators on H such that

$$\Lambda_t \rho = V(t)\rho V^+(t), \qquad \rho \in L^1(H)$$

The infinitesimal generator h of the group $V(t)$ is the Hamiltonian of the system Σ. In this case of Hamiltonian dynamics the answer about the existence of

$$w_1 - \lim_{t \to \infty} \Lambda_t \rho$$

gives the following theorem:

Theorem (Prugovečki and Tip, 1974). Let $\rho \in L^1_+(H)$. Then

$$w_1 - \lim_{t \to \infty} \Lambda_t \rho$$

exists iff ρ is invariant, i.e., $\Lambda_t \rho = \rho$ for every $t \geqslant 0$.

(b) For the dynamical semigroups the function $t \to \Lambda_t \rho$ has continuous derivatives of all orders with respect to t for ρ in the dense subset D of $L^1(H)$ (Hille and Philips, 1957). From the linearity and continuity of the function Tr we get that the mapping $t \to \mathrm{Tr}\,\{(\Lambda_t \rho)a\}$ has continuous derivatives of all orders for every $\rho \in D$ and $a \in O(H)$. Thus

$$w_1 - \lim_{t \to t_0} \Lambda_t \rho = \rho_0$$

(where ρ_0 is invariant state) can exist only for $t_0 = \infty$—the dynamical evolution becomes stationary exclusively at infinity.

(B) The situation is radically changed in the case of an arbitrary (also nonlinear) dynamical map. We give a simple example concerning the spin-$\frac{1}{2}$ system. The space H is now a two-dimensional complex Hilbert space. The arbitrary density matrix ρ of the system can be written in the form

$$\rho = \sigma_0 + \sum_{i=1}^{3} \sigma_i x_i = \sigma_0 + \mathbf{x} \cdot \boldsymbol{\sigma}$$

where $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ are the Pauli operators; $\sigma_0 = \frac{1}{2}\,\mathbb{1}$ and $x_i \in \mathbb{R}^1$ are such that

$$\sum_{i=1}^{3} x_i^2 \leqslant 1$$

Let Λ map $\mathscr{S}(H) \to \mathscr{S}(H)$. Then it induces a transformation $\tilde{\Lambda}$ of the unit ball K in $\mathbb{R}^3$ into itself:

$$\Lambda \rho = \sigma_0 + \tilde{\Lambda}\mathbf{x} \cdot \boldsymbol{\sigma}$$

whenever $\rho = \sigma_0 + \mathbf{x} \cdot \boldsymbol{\sigma}$. We have the following theorem (Guz, 1974): The family of transformations Λ_t: $\mathscr{S}(H) \to \mathscr{S}(H)$ gives a dynamical evolution of the spin-$\frac{1}{2}$ system iff the induced transformations $\tilde{\Lambda}_t$: $K \to K$ satisfy the following conditions: (i) $\tilde{\Lambda}_0 \mathrm{x} = \mathrm{x}$ for all $\mathrm{x} \in K$. (ii) For each $\mathrm{x} \in K$ the function $t \to \tilde{\Lambda}_t \mathrm{x}$, $t \geqslant 0$, is weakly continuous in $\mathbb{R}^3$.

Using the Theorem we can give the following example: Let

$$\tilde{\Lambda}_t \mathrm{x} = [e^{-\lambda t} x_1, f_2(t) \cdot x_2, f_3(t, x_3)]$$

where $\lambda > 0$; f_2, f_3 are continuous, real functions of $t \geqslant 0$, such that $f_2(0) = 1$, $f_3(0, x) = x$ for each $x \in [0, 1]$ and

$$e^{-2\lambda t} x_1^2 + f_2^2(t) x_2^2 + f_3^2(t, x_3) \leqslant 1$$

for $t \geqslant 0$ and $x_1, x_2, x_3 \in \mathbb{R}^1$ performing the inequality

$$\sum_{i=1}^{3} x_i^2 \leqslant 1$$

So defined $\tilde{\Lambda}_t$ induces a dynamical map Λ_t for our system. We take functions f_2 and f_3 such that: $f_2(t) = \gamma_2$ for $t \geqslant t_2$ and $f_3(t, x) = \gamma_3(x)$ for $t \geqslant t_3$, $x \in [0, 1]$ $t_2, t_3 < \infty$.

(a) Now if the initial state ρ is generated by the $\mathbb{R}^3$ vector of the form $(0, x_2, x_3)$ then $\Lambda_t \rho = \rho(t) = \text{const}$ for $t \geqslant \max(t_2, t_3)$ and we have stationary evolution after this time.

(b) If the initial state is of the form $\rho = \sigma_0 + x_2 \sigma_2$ then we have the linear time evolution but not of the semigroup type if $f_2(t) \neq e^{\lambda_2 t}$. The process becomes stationary after the finite time t_2.

(c) If the initial state is of the form $\rho = \sigma_0 + x_1 \sigma_1$ then we have the dynamical evolution of the semigroup type and there are true inferences of (A).

Conclusions

From the analysis of real physical systems such as the harmonic oscillator in a heat bath (Davies, 1973) or the system in time-dependent external field we see that the time evolution need not be generated by a linear dynamical semigroup. We get the semigroups only after some limiting procedure (in the oscillator case, in the weak coupling limit). Therefore when we are dealing with quantum systems in arbitrary external potentials we have various (finite or infinite) forms of the time asymptotics. In this case the existence or nonexistence of the stationary evolution for some class of initial states may be an argument for Markovian or non-Markovian character of motion.

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